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The Magnetohydrodynamics of Convection-Dominated Accretion Flows

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ABSTRACT

Radiatively inefficient accretion flows onto black holes are unstable due to both an outwardly decreasing entropy ("convection") and an outwardly decreasing rotation rate (the "magnetorotational instability"; MRI). Using a linear magnetohydrodynamic stability analysis, we show that long-wavelength modes with $\lambda/H \gg \beta^{-1/2}$ are primarily destabilized by the entropy gradient and that such "convective" modes transport angular momentum *inwards* (λ is the wavelength of the mode, H is the scale height of the disk, and β is the ratio of the gas pressure to the magnetic pressure). Moreover, the stability criteria for the convective modes are the standard Høiland criteria of hydrodynamics. By contrast, shorter wavelength modes with $\lambda/H \sim \beta^{-1/2}$ are primarily destabilized by magnetic tension

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and differential rotation. These "MRI" modes transport angular momentum *outwards.* The convection-dominated accretion flow (CDAF) model, which has been proposed for radiatively inefficient accretion onto a black hole, posits that inward angular momentum transport and outward energy transport by long-wavelength convective fluctuations are crucial for determining the structure of the accretion flow. Our analysis suggests that the CDAF model is applicable to a magnetohydrodynamic accretion flow provided the magnetic field saturates at a sufficiently sub-equipartition value ($\beta \gg 1$), so that long-wavelength convective fluctuations with $\lambda/H \gg \beta^{-1/2}$ can fit inside the accretion disk. Numerical magnetohydrodynamic simulations are required to determine whether such a sub-equipartition field is in fact obtained.

Subject Headings: accretion — accretion disks — black hole physics — convection — instabilities —MHD — turbulence

1. Introduction

Models of radiatively inefficient accretion flows provide a useful framework for interpreting observations of low-luminosity black hole X-ray binaries and active galactic nuclei (see, e.g., Narayan, Mahadevan & Quataert 1998; Quataert 2001; Narayan 2002 for reviews). In the past few years, there has been rapid theoretical progress in understanding the dynamics of such flows. Much of this advance has been driven by numerical simulations. In particular, a number of hydrodynamic simulations have been reported in which angular momentum transport is put in "by hand" using an α prescription (e.g., two-dimensional simulations by Igumenshchev, Chen, & Abramowicz 1996; Igumenshchev & Abramowicz 1999, 2000; Stone, Pringle, & Begelman 1999; and three-dimensional simulations by Igumenshchev, Abramowicz & Narayan 2000). Such simulations reveal a flow structure very different from advection-dominated accretion flow models (ADAFs) that were proposed to describe the structure of radiatively inefficient accretion flows (Ichimaru 1977; Narayan & Yi 1994; Abramowicz et al. 1995). In an ADAF, the gas accretes rapidly, and the gravitational potential energy of the accreting gas is stored as thermal energy and advected into the central black hole. By contrast, in the hydrodynamic simulations, the rate of mass accretion is much smaller and strong radial convection efficiently transports energy outwards. The simulations have been interpreted in terms of a "convection-dominated accretion flow" model (CDAF; Narayan, Igumenshchev & Abramowicz 2000; Quataert & Gruzinov 2000; Abramowicz et al. 2002). In a CDAF, convection simultaneously transports energy outwards and angular momentum inwards, strongly suppressing the accretion rate onto the central black hole.

The relevance of the hydrodynamic simulations, and the CDAF model derived from them, is unclear. This is primarily because of the *ad hoc* treatment of angular momentum transport. It is believed that angular momentum transport in accretion flows is primarily due to MHD turbulence initiated by the magnetorotational instability (MRI; Balbus & Hawley 1991; BH91). Balbus & Hawley have argued that, because of the fundamental role played by magnetic fields, hydrodynamic models typically cannot describe the structure of the accretion flow (or differentially rotating systems more generally; e.g., Balbus & Hawley 1998; Balbus 2000; 2001). They have applied this criticism in detail to the CDAF model (Hawley, Balbus, & Stone 2001; Balbus & Hawley 2002; hereafter BH02). In an independent argument, BH02 also suggest that CDAF models violate the second law of thermodynamics. We do not consider this issue here, but intend to deal with it in a separate paper.

In this paper we discuss the physics of the CDAF model within the framework of MHD. Apart from clarifying the theoretical underpinnings of the CDAF model, we believe that our analysis also provides a useful illustration of the conditions under which a hydrodynamic model is applicable to an intrinsically magnetohydrodynamic situation; this is important in other astrophysical contexts, e.g., the excitation of density waves at Lindblad resonances in a magnetized accretion flow, or diskoseismology models of QPOs.

The paper is organized as follows. In §2 we discuss the linear stability of a differentially rotating and thermally stratified plasma in the presence of a weak vertical magnetic field. We identify and explain the difference between "convective" modes and "MRI" modes and show that the former transport angular momentum inwards while the latter usually transport angular momentum outwards. Then, in §2.1 we generalize this result to an arbitrary field orientation and arbitrary stratification. In §3 we discuss the implications of our results. In particular, we use the linear analysis to speculate about the conditions under which the saturated nonlinear turbulence in an accretion flow might give rise to a CDAF-like structure. In §4 we summarize our results.

2. Convective Modes in a Magnetized Differentially Rotating Plasma

We consider the linear stability of a differentially rotating and thermally stratified plasma in the presence of a weak magnetic field. We specialize to the case of a vertical magnetic field, $B_z \equiv B$, and modes with purely vertical wavevectors, $k_z \equiv k$. We also take the rotation rate, pressure, and density to be constant on cylinders, i.e., $\Omega(R)$, $P(R)$, and $\rho(R)$. We thus employ the same equations as those in BH02 (which are a simplification of the more general analysis in BH91). Although the problem we analyze is identical to that considered by BH02, we differ in the interpretation of the results. In particular, we identify

a set of modes that have all the features of growing convective modes in hydrodynamics, including inward transport of angular momentum.

The thermal stratification of the flow can be described by the Brunt-Väisälä frequency,

$$N^2 = -N_R^2 = \frac{3}{5\rho}\frac{\partial P}{\partial R}\frac{\partial \ln P\rho^{-5/3}}{\partial R}. \tag{1}$$

In the absence of rotation, the system is convectively unstable if $N^2 > 0$. In a rotating medium, but ignoring magnetic fields, convection is present only if $N^2 > \kappa^2$, where $\kappa^2 = 4\Omega^2 + d\Omega^2/d\ln R$ is the epicyclic frequency.[2] Note that $N^2 > \kappa^2$ requires a sound speed comparable to the rotation speed.

For a weakly magnetized medium, BH91 derived the behavior of linear perturbations including the effects of thermal stratification. For fluid displacements of the form $\xi \propto \exp[ikz + \gamma t]$, and for the particular geometry considered here, the growth rate γ is given by

$$\gamma^2 = -(kv_A)^2 + \frac{1}{2}\left[N^2 - \kappa^2 \pm \sqrt{(N^2-\kappa^2)^2 + 16\Omega^2(kv_A)^2}\right], \tag{2}$$

where $v_A = B/\sqrt{4\pi\rho}$ is the Alfvén speed. For the remainder of this section we focus on the growing mode in equation (2); this has $\gamma^2 > 0$ and corresponds to the positive sign of the square root term. In the Appendix we present a more general analysis of both growing and oscillatory modes that helps clarify some of the issues discussed in this section.

To understand the physics behind the dispersion relation in equation (2) consider the equations for the radial and azimuthal displacements of fluid elements (BH91):

$$\frac{\partial^2 \xi_R}{\partial t^2} - 2\Omega\frac{\partial \xi_\phi}{\partial t} = -\left(\frac{d\Omega^2}{d\ln R} + (kv_A)^2\right)\xi_R + N^2\xi_R, \tag{3}$$

$$\frac{\partial^2 \xi_\phi}{\partial t^2} + 2\Omega\frac{\partial \xi_R}{\partial t} = -(kv_A)^2\xi_\phi. \tag{4}$$

Solving equation (4) for ξ_ϕ we can substitute into equation (3) to find

$$\frac{\partial^2 \xi_R}{\partial t^2} = a_H + a_{M,s} + a_{M,d}, \tag{5}$$

where the three acceleration terms are of the form

$$a_H = (N^2 - \kappa^2)\xi_R, \qquad a_{M,s} = -(kv_A)^2\xi_R, \qquad a_{M,d} = \left[\frac{4\Omega^2 k^2 v_A^2}{\gamma^2 + k^2 v_A^2}\right]\xi_R. \tag{6}$$

[2] The requirements for convection in an unmagnetized rotating medium are the Høiland criteria (e.g., Tassoul 1978). These reduce to the condition $N^2 > \kappa^2$ for the case considered here. For simplicity, we will refer to this condition as the Høiland criterion in this section. §2.1 treats the general case.

Since $\partial^2 \xi_R / \partial^2 t = \gamma^2 \xi_R$ it is straightforward to solve equation (5) and obtain the dispersion relation in equation (2).

The first acceleration term in equation (5), a_H, is a pure "hydrodynamic" or "Høiland" term (hence the subscript H) which is present even in the absence of magnetic fields. This term can be either stabilizing or destabilizing; it is stabilizing when $N^2 < \kappa^2$ (the medium is convectively stable by the Høiland criteria) and destabilizing when $N^2 > \kappa^2$ (convectively unstable). The second and third terms in equation (6) owe their existence to the magnetic field (hence subscript M). One of them is always stabilizing (subscript s) and the other is always destabilizing (subscript d).

Equations (3)–(6) have a very clean physical interpretation, as we explain in the rest of this section. For the simple geometry considered here (vertical field and vertical wavevector), gas and magnetic pressure forces are unimportant. If $N^2 = 0$, the only forces in the problem are magnetic tension and rotating frame dynamics (e.g. the Coriolis force). Equations (3) and (4) then describe the behavior of fluid elements that are coupled by the tension of magnetic field lines. As Balbus & Hawley (1992, 1998) have discussed, these equations are identical to those describing fluid elements coupled by a spring of spring constant $(kv_A)^2$. For a stratified medium with $N^2 \neq 0$, the radial acceleration of a fluid element has an additional component due to the radial buoyancy force. This is the $N^2 \xi_R$ term in equations (3) and (5). When $N^2 < \kappa^2$, this term does not have a large effect and introduces modest changes to the values of numerical factors. However, when $N^2 > \kappa^2$, which corresponds to a convectively unstable system by the Høiland criterion, the acceleration term a_H becomes positive and destabilizing. This introduces qualitatively new effects.

The angular momentum flux of a linear mode is given by $R\Omega T_{R\phi}$, where $T_{R\phi}$ is the $R-\phi$ component of the fluid stress tensor,

$$T_{R\phi} = \rho(\delta v_R \delta v_\phi - \delta v_{AR} \delta v_{A\phi}), \tag{7}$$

$\delta\mathbf{v}$ is the perturbed fluid velocity, and $\delta\mathbf{v_A} \equiv \delta\mathbf{B}/\sqrt{4\pi\rho}$. Using the eigenfunctions in BH91 we confirm BH02's expression for the stress tensor, namely

$$\frac{T_{R\phi}}{\rho|\delta v_R^2|} = \frac{\Omega}{D\gamma}\left[\frac{k^2 v_A^2}{\gamma^2}\left(\left|\frac{d\ln\Omega}{d\ln R}\right| + 2\right) - \frac{\kappa^2}{2\Omega^2}\right] \equiv \left(\frac{\Omega}{\gamma}\right) t_{R\phi}, \tag{8}$$

where $D = 1 + k^2 v_A^2/\gamma^2$ and $t_{R\phi}$ is a useful dimensionless stress whose sign gives the sign of $T_{R\phi}$.

We are now in a position to elucidate the physics of the linear instability calculation. Figure 1a shows a plot of the growth rate γ/Ω as a function of the dimensionless wavevector kv_A/Ω for 5 values of the Brunt-Väisälä frequency: $N^2/\Omega^2 = 0, 0.5, 1, 1.5$, and 2. We have

assumed a Keplerian rotation profile, for which $\kappa^2 = \Omega^2$. Thus, the first two choices of N^2 correspond to a stable entropy gradient by the Høiland criterion, the third is neutrally stable, and the last two choices correspond to an unstable entropy gradient. Figure 1b shows the dimensionless stress $t_{R\phi}$ as a function of kv_A/Ω for the same five values of N^2. It is useful to note that the self-similar ADAF model gives $N^2/\Omega^2 = 15(\gamma_{ad} - 1)/4\gamma_{ad}$, where γ_{ad} is the adiabatic index of the gas (Narayan & Yi 1994).[3] For example, if $\gamma_{ad} = 1.5$, $N^2/\Omega^2 = 1.25$.

For $N^2 = 0$, the results reproduce the usual MRI. The growth rate is very small for $kv_A \ll \Omega$ and peaks when kv_A is $\approx \Omega$, with a peak value of $\gamma = 3\Omega/4$; the angular momentum flux is always outwards ($T_{R\phi} > 0$). The instability is clearly triggered by the magnetic field via the destabilizing term $a_{M,d}$, since the other two terms in equation (5) are both stabilizing (negative). The MRI survives with qualitatively the same behavior even for non-zero N^2, so long as the Høiland criterion for convective stability ($N^2 < \kappa^2$) is satisfied.

Consider, however, a flow with $N^2 > \kappa^2$. Such a flow is unstable to convection according to hydrodynamics, and is therefore the case of interest for CDAFs. Figure 1 shows that, for $N^2 > \kappa^2$, modes with $kv_A \sim \Omega$ are not very different from their $N^2 = 0$ counterparts. The angular momentum flux is still outwards and the growth rate of the mode is comparable (though somewhat larger). By contrast, long-wavelength modes are very different. As Figure 1 shows, long-wavelength modes are strongly unstable and they transport angular momentum *inwards*. In fact, as can be seen by inspection, the $kv_A \ll \Omega$ limit of the dispersion relation in equation (2) (with the + root) gives $\gamma^2 = N^2 - \kappa^2$, and the corresponding stress tensor is $T_{R\phi}/(\rho|\delta v_R|^2) = -\kappa^2/[2\Omega(N^2 - \kappa^2)^{1/2}] < 0$. The long-wavelength modes are clearly independent of the magnetic field and behave very differently from MRI modes.

The physics of the $kv_A \ll \Omega$ modes is as follows. In the pure MRI problem, namely $N^2 = 0$ (or more generally $N^2 < \kappa^2$), long-wavelength modes are only weakly unstable because the magnetic tension forces that are central to the MRI are weak; the field lines are hardly bent by a long-wavelength perturbation. By contrast, when $N^2 > \kappa^2$, the fluid has two destabilizing forces, the buoyancy term (a_H) due to an unstable entropy gradient and the standard MRI term ($a_{M,d}$). The buoyancy force that drives convection is independent of the wavelength of the perturbation, whereas the MRI depends on k. For long wavelengths, buoyancy is much more important and controls all the physics of the mode, both the growth rate and the angular momentum transport. Moreover, we show in the Appendix that the "convective" mode driven by buoyancy is the *only* unstable long-wavelength mode in MHD for a Høiland-unstable medium ($N^2 > \kappa^2$). In particular, long-wavelength perturbations for which magnetic tension dominates (referred to as − modes in the Appendix) are stable

[3]This expression is not valid if $\gamma_{ad} = 5/3$ since then $\Omega^2 = 0$ in the analytical models.

oscillatory waves rather than unstable modes (see eq. A3 and Fig. 3).

Figure 2 explicitly compares the relative importance of the two destabilizing terms, the buoyancy term a_H and the magnetic tension term $a_{M,d}$. Let us identify a mode as an "MRI" mode if $a_H \lesssim a_{M,d}$ and as a "convective" mode if $a_H > a_{M,d}$. With this natural identification, we see that, for $N^2 < \kappa^2$, all unstable modes are MRI modes, regardless of the wavelength. Even when $N^2 > \kappa^2$, short-wavelength modes with $kv_A \gtrsim \Omega$ are still MRI modes. However, unstable long-wavelength modes with $kv_A \ll \Omega$ are clearly convective modes. The growth of these modes is due entirely to the unstable buoyancy force; correspondingly, the modes transport angular momentum inwards (Fig. 1b), just as indicated by hydrodynamic studies (e.g., Ryu & Goodman 1992). Indeed these modes are virtually indistinguishable from their hydrodynamic counterparts. *All of the properties of hydrodynamic convective instabilities therefore carry over to a magnetohydrodynamic analysis.*

2.1. The Hydrodynamic Limit

Magnetic tension should be weak for fluctuations with $kv_A \ll \Omega$ regardless of the details of the magnetic field geometry and thermal stratification. In this subsection, we show this explicitly by considering the long-wavelength limit of the most general axisymmetric MHD dispersion relation.

Balbus (1995) considered the linear, adiabatic, and axisymmetric stability of a differentially rotating weakly magnetized plasma in the presence of both vertical and radial stratification. He showed that the dispersion relation is given by

$$\tilde{\omega}^4 \frac{k^2}{k_z^2} - \tilde{\omega}^2 \left[\frac{3}{5\rho}(DP)(D \ln[P\rho^{-5/3}]) + \frac{1}{R^3} D(R^4\Omega^2) \right] - 4\Omega^2 (\mathbf{k}\cdot\mathbf{v_A})^2 = 0, \tag{9}$$

where

$$D = \left(\frac{k_R}{k_z}\frac{\partial}{\partial z} - \frac{\partial}{\partial R} \right), \tag{10}$$

and $\tilde{\omega}^2 = -\gamma^2 - (\mathbf{k}\cdot\mathbf{v_A})^2$ (for fluctuations $\propto \exp[\gamma t + i\mathbf{k}\cdot\mathbf{r}]$). Unlike in the previous section, Ω, P, and ρ are now allowed to be functions of both z and R, the magnetic field has an arbitrary direction (i.e., B_R, B_ϕ, and B_z components), and the wavevector is given by $\mathbf{k} = k_R\hat{\mathbf{R}} + k_z\hat{\mathbf{z}}$.

For $\mathbf{k}\cdot\mathbf{v_A} = 0$ (but k/k_z, k_R/k_z, etc., finite), equation (9) reduces to

$$\gamma^2 \frac{k^2}{k_z^2} + \left[\frac{3}{5\rho}(DP)(D \ln[P\rho^{-5/3}]) + \frac{1}{R^3} D(R^4\Omega^2) \right] = 0. \tag{11}$$

Equation (11) is independent of the magnetic field. Indeed, as noted by Balbus (1995), it describes the linear, adiabatic, axisymmetric, hydrodynamic stability of a differentially rotating and thermally stratified medium (e.g., Goldreich & Shubert 1967). As is readily confirmed, equation (11) has unstable solutions if and only if the Høiland criteria are satisfied (e.g., Tassoul 1978).

The hydrodynamic limit in equation (11) formally requires setting $\mathbf{k} \cdot \mathbf{v_A} = 0$ in the MHD dispersion relation (eq. [9]). If $\mathbf{k} \cdot \mathbf{v_A} \neq 0$, the full dispersion relation (eq. [9]), which has twice the number of modes (because it is fourth order in γ rather than second order), must be used. However, if the medium is unstable by the Høiland criteria, and if we focus on long-wavelength MHD instabilities with $\mathbf{k} \cdot \mathbf{v_A} \ll \Omega$, then the MHD system will behave like its hydrodynamic counterpart; in particular, the only unstable mode is effectively hydrodynamical and its growth rate is given by equation (11) with small corrections $\sim (\mathbf{k} \cdot \mathbf{v_A}/\Omega)^2 \ll 1$. Note that equation (11) describes the unstable growing mode only when the medium is Høiland unstable. If the medium is Høiland stable, the only possible instabilities are intrinsically MHD in nature and equation (9) must be used even for $\mathbf{k} \cdot \mathbf{v_A} \ll \Omega$. This point is explained in more detail in the Appendix.

The analysis in this section shows that all of the properties of axisymmetric hydrodynamic instabilities carry over to MHD in the long-wavelength limit. In particular, the results of the previous section are general, and are not an artifact of the simplifying assumptions made there. Although our quantitative analysis is restricted to axisymmetry, we suspect that long-wavelength non-axisymmetric modes will behave hydrodynamically as well (since they also do not significantly bend the magnetic field lines).

3. Implications for CDAFs

To assess the implications of our linear analysis it is useful to consider wavelengths relative to the scale height of the disk, $H \approx c_s/\Omega$, where c_s is the sound speed. The maximally growing MRI mode has $kH \sim \beta^{1/2}$, where $\beta \approx c_s^2/v_A^2$ is the ratio of the thermal energy to the magnetic energy in the disk. By contrast, the longer wavelength convective modes have $kH \ll \beta^{1/2}$; more quantitatively, the linear analysis suggests that buoyancy forces become important for $kH \lesssim 0.3\beta^{1/2}$ (Fig. 2). For these modes to be of interest they must fit in the disk and so must have $kH \gtrsim 1$. This in turn requires $\beta \gg 1$, i.e., that the magnetic field must be sub-equipartition ($\beta \gtrsim 10$ may suffice).

Given the above analysis, we propose the following identification between the hydrodynamic CDAF model and the present MHD results. If the magnetic field saturates at a

sub-equipartition value, the most unstable MRI mode operates on a scale $\sim \beta^{-1/2}H \ll H$ and transports angular momentum outwards. Moreover, if the medium is Høiland unstable, *only* such short-wavelength fluctuations transport angular momentum outwards. Longer wavelength convective fluctuations are dominated by scales $\sim H$ and transport angular momentum inwards (Fig. 1).

Which fluctuations are more important? Or, more precisely, which part of the power spectrum of fluctuations is more important, those with $kH \sim 1$ or those with $kH \sim \beta^{-1/2}$? This cannot really be addressed analytically because we do not understand the saturation of the linear instabilities (and it is clear that they are coupled; e.g., convection, by stirring the fluid, can help to build up the magnetic field). Nonetheless, it is typically the case that the energy in a turbulent plasma is dominated by the largest unstable scales in the medium. This suggests that, if $\beta \gg 1$ (perhaps $\beta \gtrsim 10$ is sufficient), the larger-scale convective fluctuations will dominate the dynamics, as proposed in the hydrodynamic models. Moreover, the inward angular momentum transport by convective modes depends on the magnitude of $N^2 - \kappa^2$ (Fig. 1b). Thus there is a natural way for the inward convective transport to self-adjust to counteract outward transport by small scale MRI fluctuations. In the limit of efficient convection and $\beta \gg 1$ we would expect marginal stability to convection according to the Høiland criteria, just as in the hydrodynamic models (Quataert & Gruzinov 2000).

It is important to stress several points:

(1) Although we believe that our linear stability analysis sheds important light on the physics of radiatively inefficient accretion flows, it does not directly address the fully developed nonlinear turbulence that is of primary importance to the accretion flow structure. This requires numerical simulations.

(2) The picture proposed here is very similar to the hydrodynamic models of CDAFs; the primary change is that we have theoretical support from a magnetohydrodynamic analysis.

(3) Balbus (2001) has shown that magnetized dilute plasmas can be convectively unstable in the presence of an outwardly decreasing temperature, rather than just an outwardly decreasing entropy. It would be interesting to incorporate this into future work.

(4) The extent to which one can usefully distinguish scales in the power spectrum that are convective from those that are magnetorotational depends on β, and becomes more and more useful if $\beta \gg 1$. It would be straightforward if the convective and MRI modes occupied distinct and well-separated regions of wavevector space. Then even the nonlinear physics of the instabilities could be distinct. To cite an example from plasma physics, heat transport in fusion devices is due to instabilities on both proton and electron Larmor radii scales and one can often distinguish the physics of each mechanism separately even in the nonlinear

regime (e.g., Dorland et al. 2000; Rogers, Dorland & Kotschenreuther 2000). As Figure 1 shows, such a separation in wavevector space is not present for our problem. On the other hand, Figs. 1 and 2 do show quite clearly that the physics of the convective modes is very different from that of the MRI modes.

(5) For a Høiland-unstable medium ($N^2 > \kappa^2$), there is a nice continuity between the hydrodynamic limit and the strong field MHD limit. In wavevector space, the field-affected modes (the MRI modes) are restricted to larger values of k where magnetic tension is important. For very weak fields, these modes are at very large k, and most of the long-wavelength modes that probably dominate the dynamics are convective. As the field strength increases, however, the MRI modes expand downwards in k space and become more and more important. Another way to look at this is as follows. When the fluid is unstable according to the Høiland criterion, the switch that determines whether or not convection is important is not the presence or absence of a magnetic field, but rather the magnitude of the dimensionless parameter $kv_A/\Omega \approx kH/\beta^{1/2}$. When kv_A/Ω is small, convection dominates the physics, whereas when it is $\gtrsim 1$, the magnetic field dominates.

4. Conclusion

In this paper we have used linear stability theory to assess the conditions under which a hydrodynamic analysis is appropriate for an intrinsically magnetohydrodynamic accretion flow. We have shown that, when the medium is unstable by the Høiland criteria, long-wavelength instabilities with $kv_A \ll \Omega$ are effectively hydrodynamical. Physically, this is because magnetic tension is unimportant for sufficiently long-wavelength perturbations. Formally, the dispersion relation for a magnetohydrodynamic plasma reduces in this limit to that of hydrodynamics (§2). This shows that all of the properties of hydrodynamic instabilities carry over to an MHD analysis in the long-wavelength limit.

For the particular case of radiatively inefficient accretion flows that are unstable to both convection and the MRI, the relevant parameter that distinguishes convective behavior from magnetorotational behavior is the quantity $kv_A/\Omega \approx H/(\lambda\beta^{1/2})$. Long-wavelength modes for which $\lambda/H \gg \beta^{-1/2}$ are convective, while modes for which $\lambda/H \sim \beta^{-1/2}$ are magnetorotational. The convective modes are primarily driven by buoyancy, not magnetic tension (Fig. 2). Moreover, they transport angular momentum *inwards* rather than outwards (Fig. 1b), just as indicated by hydrodynamic studies. For the long-wavelength modes to be of interest the accretion flow must have $\beta \gg 1$ so that the convective fluctuations fit inside the disk. Under these conditions we believe that the hydrodynamic CDAF models can accurately describe the structure of the accretion flow. The precise β that allows a

hydrodynamical analysis is uncertain and can be determined only by magnetohydrodynamic simulations (the linear analysis suggests that $\beta \gtrsim 10$ may suffice).

It is interesting to note that the saturation β of the MRI does not appear to be universal. Given the plausible dependence on β highlighted above, this may have important implications for the structure of radiatively inefficient accretion flows. Three-dimensional MHD simulations find that that if the field is initially toroidal a radiatively inefficient accretion flow develops into a CDAF-like state (Igumenshchev, Abramowicz & Narayan 2002), while if the field has a vertical component the structure is very different (Hawley, Balbus & Stone 2001; Hawley & Balbus 2002; see also Stone & Pringle's 2001 2D simulations).

To conclude, it is important to emphasize that the primary conclusion of the analytic arguments in this paper is not that CDAFs *actually do* describe the structure of radiatively inefficient accretion flows, but rather that they *could.* In particular, our linear MHD analysis supports all of the key CDAF assumptions so long as $\beta \gg 1$.

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A. Appendix

In the main text we focused exclusively on unstable modes and therefore considered only the positive sign of the root in equation (2). Here we generalize the discussion and consider both unstable and stable modes. To this end, we rewrite equation (2) as follows

$$\gamma^2 = -(kv_A)^2 + \frac{(N^2-\kappa^2)}{2}\left[1 \pm \sqrt{1+16\Omega^2(kv_A)^2/(N^2-\kappa^2)^2}\right]. \tag{A1}$$

The $+$ and $-$ signs in equation (A1) map directly onto the $+$ and $-$ signs in equation (2) when $N^2-\kappa^2 > 0$, but the mapping is reversed when $N^2-\kappa^2 < 0$. Equation (A1) is, of course, completely equivalent to equation (2), but for this Appendix we find it more convenient to work with the $\pm$ convention in equation (A1) because of its behavior when $N^2-\kappa^2$ changes sign. In what follows, we refer to the $+$ sign in equation (A1) as the "$+$ mode" and the $-$ sign as the "$-$ mode."

Let us focus on the long-wavelength limit $kv_A/\Omega \ll 1$. The growth rates of the $+$ and $-$ modes are then given by

$$\gamma_+^2 \approx (N^2-\kappa^2) + \left(\frac{4\Omega^2}{N^2-\kappa^2} - 1\right)(kv_A)^2, \tag{A2}$$

$$\gamma_-^2 \approx -\left(\frac{4\Omega^2}{N^2-\kappa^2} + 1\right)(kv_A)^2. \tag{A3}$$

In the limit $kv_A/\Omega \ll 1$, the growth rate of the $+$ mode is the same as in hydrodynamics: $\gamma_+^2 = N^2-\kappa^2$. If $N^2 > \kappa^2$ (Høiland unstable), γ_+^2 is positive and the mode is unstable, while if $N^2 < \kappa^2$, γ_+^2 is negative and the mode is oscillatory in nature. In either case, the mode is completely insensitive to the magnetic field.

The $-$ mode, on the other hand, has a growth rate that vanishes as $kv_A \to 0$, which is a clear indication that this mode is strongly influenced by the magnetic field. Indeed, the mode owes its very existence to the presence of the magnetic field. The $-$ mode is associated with the classic MRI of BH91.

An inspection of equations (A2) and (A3) reveals the following interesting result: assuming that $|N^2-\kappa^2| < 4\Omega^2$ (which is almost always the case), only one of the two modes can be unstable. When the medium is convectively stable ($N^2 < \kappa^2$), the $+$ mode (which we argued above is hydrodynamic in nature) is obviously stable. In this case, the $-$ mode is unstable and gives the long-wavelength limit of the MRI. However, when the medium is convectively unstable ($N^2 > \kappa^2$), then it is the $+$ mode that grows, while the $-$ mode is

stable. Thus, in the long wavelength limit, the MRI actually disappears if the medium is unstable by the Høiland criterion. Fortunately, this is precisely when the medium is convectively unstable so there are always unstable modes available, regardless of the sign of $N^2 - \kappa^2$.

Figure (3) shows γ^2 as a function of $(kv_A)^2$ for the $+$ and $-$ modes for representative values of N^2/Ω^2. We see that modes that are stable ($\gamma^2 < 0$) in the limit $kv_A \to 0$ remain stable for all values of k, while modes that are unstable at small values of kv_A reach a maximum growth rate for $kv_A \sim \Omega$ and then become stable with increasing kv_A.

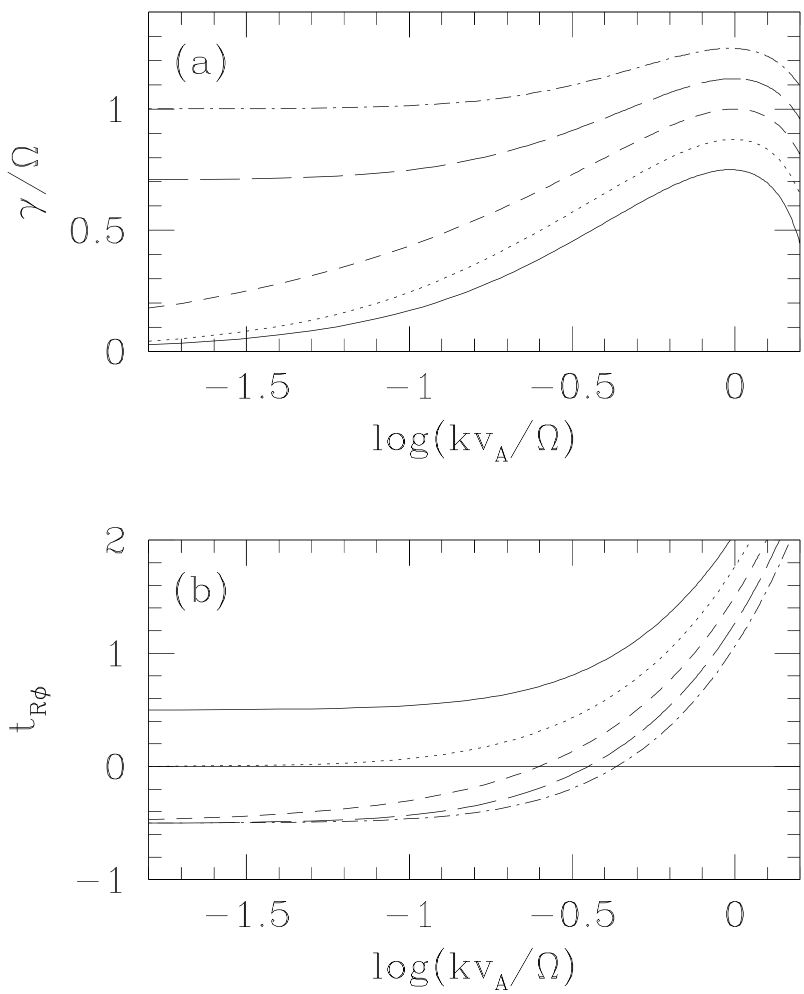

Fig. 1.— (a) Dimensionless growth rate γ/Ω as a function of dimensionless wavevector kv_A/Ω in a Keplerian system ($\kappa^2 = \Omega^2$), for five choices of the Brunt-Väisälä frequency: $N^2/\Omega^2 = 0$ (solid line), 0.5 (dotted line), 1 (short-dashed line), 1.5 (long-dashed line), 2 (dot-dashed line). The system is unstable by the Høiland criterion for $N^2/\Omega^2 = 1.5$ and 2. For these two cases, the growth rate remains finite in the long-wavelength limit $kv_A/\Omega \ll 1$ and the modes are indistinguishable from the corresponding unstable hydrodynamical modes. (b) Dimensionless shear stress corresponding to the same modes as in (a). For $N^2/\Omega^2 = 1.5$, 2, and $kv_A/\Omega \ll 1$, the stress is negative, implying that these convective modes transfer angular momentum inwards, just as in hydrodynamics.

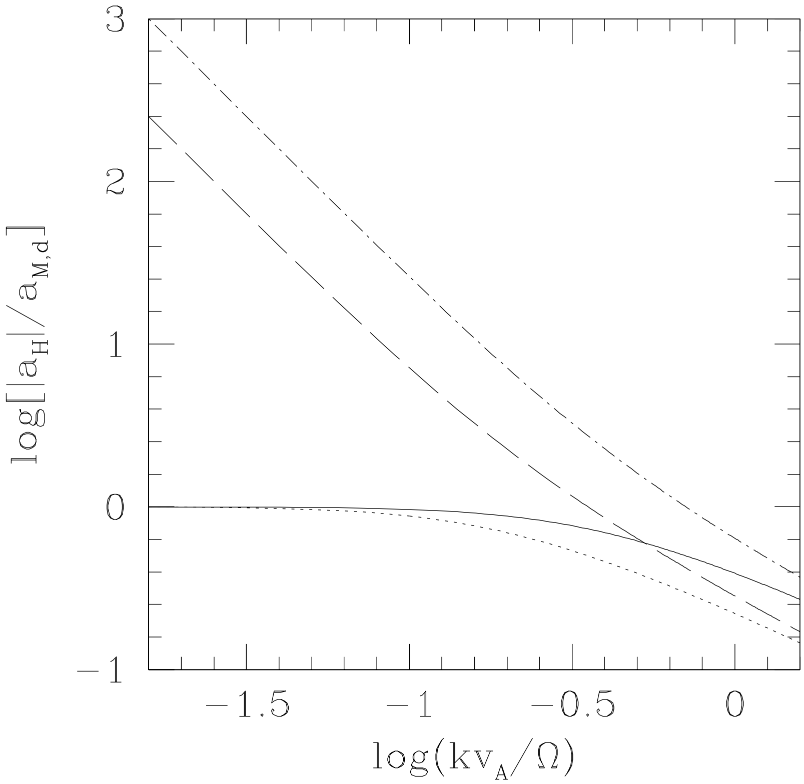

Fig. 2.— Variation of $a_H/a_{M,d}$ versus dimensionless wavevector kv_A/Ω for $N^2/\Omega^2 = 0$ (solid line), 0.5 (dotted line), 1.5 (long-dashed line), 2 (dot-dashed line). The ratio $a_H/a_{M,d}$ measures the importance of the hydrodynamic buoyancy force relative to the leading destabilizing force due to magnetic fields (see eqs. [5], [6]). For the two cases in which the system is convectively unstable according to the Høiland criterion, *viz.*, $N^2/\Omega^2 = 1.5$, 2, the buoyancy force clearly dominates at long wavelengths, confirming that these modes are primarily convective, not magnetorotational.

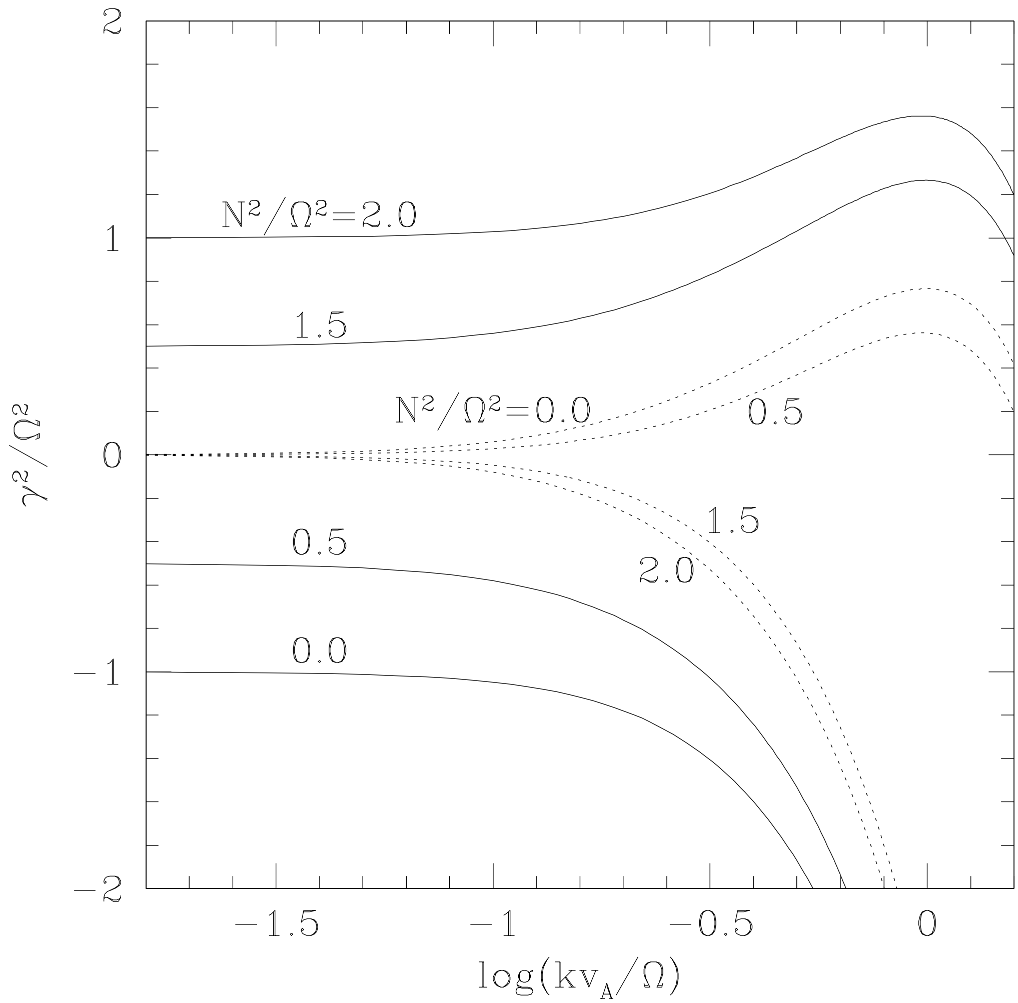

Fig. 3.— The dimensionless growth rate γ^2/Ω^2 versus dimensionless wavevector kv_A/Ω for $N^2/\Omega^2 = 0,\ 0.5,\ 1.5,\ 2$. A mode with a positive value of γ^2 is unstable, while one with a negative value of γ^2 is stable (oscillatory). The solid and dashed lines correspond to the + and − modes, respectively, in equation (A1). Each curve is labeled by the value of N^2/Ω^2.